NONSTATIONARY, PLANE, PARALLEL FLOW OF A VISCOUS ELECTRICALLY-CONDUCTING GAS WITH ANISOTROPIC CONDUCTIVITY

(NEYSTANOVIVSHEESIA PLOSKOPARALLEL'NOE TECHENIE VIAZKOGO ELEKTROPROVODNOGO GAZA S UCHETOM ANIZOTROPII PROVIDOMOSTI)

PMM Vol.26, No.5, 1962, pp. 836-841

Ia.S. UFLIAND
(Leningrad)

(Received June 4, 1962)

In this paper an investigation is made of the transient regimes which appear in the motion of an ionized, viscous gas between parallel, conducting planes in the presence of a transverse magnetic field.

Questions of magnetohydrodynamics relating to the nonstationary flow of an incompressible conducting medium have been the subject of numerous investigations in recent years. The first work along these lines was, apparently, the paper of Regirer [1], in which the transverse magnetic field is taken to be homogeneous, the walls of the channel non-conducting, and the motion is maintained by a longitudinal pressure drop. In this case, the velocity and the induced magnetic field each have only one component, in the direction of the applied pressure gradient, which depend on the transverse coordinate, and the magnetohydrodynamic equations reduce to a linear system of equations of partial derivatives. The analogous problem for the case of moving walls was investigated in [2] (see also [3], in which the channel walls are taken to be perfectly conducting). Quite recently a number of similar investigations have been published, of which we note the papers by Musin [4] and Yen and Chang [5]. The problems considered were further developed in [6] and [7], in which it was shown that there is an essential influence of the conductivity of the walls on the nonstationary motion of the type considered.

1. Statement of the problem. In all the investigations referred to above, it was assumed that the effect of the cyclotron frequency, ω, of changed particles on the mean free time, τ, between collisions is small, which made it possible to take the conductivity and viscosity to be scalar and to use the usual Ohm's law. However, for sufficiently strong magnetic fields, or for a rarefied gas, the condition $\omega\tau \ll 1$ may be violated, so that it becomes necessary to use various complicated forms of Ohm's law [8]. If it is assumed that the degree of ionization is small and that for the ions the relation $\omega_i\tau_i \ll 1$ is satisfied, i.e. neglect any slip of the ions with respect to the gas, then the coefficient of viscosity η can be taken to be scalar quantity, and Ohm's law may be introduced in the following form [8,9]:

$$\mathbf{j} + \frac{\omega_e\tau^*}{H}\mathbf{j}\times\mathbf{H} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{H}) \tag{1.1}$$

where ω_e is the electron cyclotron frequency, τ^* is the mean time between collisions of electrons with ions and neutral particles, σ is the conductivity, $\mathbf{j}$ is the current density, $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields, $\mathbf{v}$ is the velocity of the medium. Under such conditions, the problem of flow in a plane channel remains linear (compressibility effects are neglected), but due to the inclusion of the Hall current the character of the transition regime becomes complicated, primarily because of the appearance of transverse currents and fields.

In [9] the nonstationary motion of a weakly ionized gas was investigated for a plane channel, including anisotropic conductivity according to the scheme outlined above, the motion being produced by a steady, longitudinal pressure gradient.

In the present paper an analogous flow of a viscous fluid between parallel walls is investigated, with the assumption that along the x- and y-axes there are applied given pressure drops, $P_x(t)$ and $P_y(t)$, with a homogeneous magnetic field in a direction perpendicular to the walls (Fig. 1). An exact solution of the problem, obtained for walls of finite conductivity in the form of complex integrals (Section 2), transforms to a simple, real form for the case of a weakly conducting medium (Section 3). It is shown that transient regimes of this type have the character of damped oscillations, and the influence of viscosity on their form is investigated. For the flows investigated, there is a qualitative difference between those with anisotropic conductivity and the ordinary isotropic case, for which the motion of a weakly conducting medium has an aperiodic character.

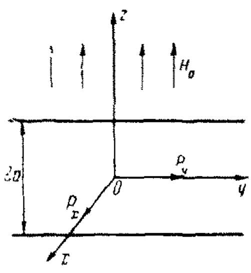

Fig. 1.

2. General solution of the problem.

With the assumptions made, the system of magnetohydrodynamic equations has the form

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] = \eta\Delta\mathbf{v} - \nabla\left(p + \frac{H^2}{8\pi}\right) + \frac{1}{4\pi}(\mathbf{H}\nabla)\mathbf{H} \tag{2.1}$$

$$4\pi\sigma\left[\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{H}\right] = 4\pi\sigma\,(\mathbf{H}\nabla)\,\mathbf{v} + \triangle\mathbf{H} + \frac{e\tau^*}{m}\,(\mathrm{rot}\ \mathbf{H}\,\nabla)\,\mathbf{H} - (\mathbf{H}\nabla)\,\mathrm{rot}\ \mathbf{H}]$$

where ρ is the density, p the pressure, e and m the charge and mass of an electron (the CGSM system is adopted, with $\mu = 1$).

It is not difficult to see that in the problem under consideration the velocity and induced magnetic field have directions parallel to the walls ($z = \pm\, a$) and depend only on the transverse coordinate z and time t. If the dimensionless quantities

$$\mathbf{u} = \frac{\mathbf{v}}{v_0}, \quad h = \frac{\mathbf{H}}{H_0}, \quad q_x = \frac{P_x a^2}{v_0\eta}, \quad q_y = \frac{P_y a^2}{v_0\eta}, \quad \zeta = \frac{z}{a}, \quad \tau = \frac{v_0 t}{a}$$
$$M = H_0 a\sqrt{\sigma/\eta}, \quad R = \frac{\rho}{\eta}\, v_0 a, \quad R_m = 4\pi\sigma v_0 a, \quad \beta = \frac{eH_0\tau^*}{m} \tag{2.2}$$

(v_0 is some characteristic velocity) are introduced, then Equations (2.1) will be satisfied if solutions $u_x(\zeta, \tau)$, $u_y(\zeta, \tau)$, $h_x(\zeta, \tau)$, $h_y(\zeta, \tau)$ can be found in the solution for the system

$$R\frac{\partial u_x}{\partial \tau} = \frac{\partial^2 u_x}{\partial \zeta^2} + \frac{M^2}{R_m}\frac{\partial h_x}{\partial \zeta} + q_x, \qquad R\frac{\partial u_y}{\partial \tau} = \frac{\partial^2 u_y}{\partial \zeta^2} + \frac{M^2}{R_m}\frac{\partial h_y}{\partial \zeta} + q_y \tag{2.3}$$
$$R_m\frac{\partial h_x}{\partial \tau} = \frac{\partial^2 h_x}{\partial \zeta^2} + \beta\frac{\partial^2 h_y}{\partial \zeta^2} + R_m\frac{\partial u_x}{\partial \zeta}, \qquad R_m\frac{\partial h_y}{\partial \tau} = \frac{\partial^2 h_y}{\partial \zeta^2} - \beta\frac{\partial^2 h_x}{\partial \zeta^2} + R_m\frac{\partial u_y}{\partial \zeta}$$

and the components of the pressure gradient are found from the relations

$$-\frac{\partial p}{\partial x} = P_x(t), \qquad -\frac{\partial p}{\partial y} = P_y(t), \qquad -\frac{\partial p}{\partial z} = \frac{1}{8\pi}\frac{\partial H^2}{\partial z} \tag{2.4}$$

For the components of the electric field we have

$$e_x = \frac{1}{R_m}\left(\beta\frac{\partial h_x}{\partial \zeta} - \frac{\partial h_y}{\partial \zeta}\right) - u_y, \qquad e_y = \frac{1}{R_m}\left(\beta\frac{\partial h_y}{\partial \zeta} + \frac{\partial h_x}{\partial \zeta}\right) + u_x$$
$$e_z = u_y h_x - u_x h_y - \frac{\beta}{2R_m}\frac{\partial h^2}{\partial \zeta}, \qquad \mathbf{e} = \frac{\mathbf{E}}{v_0 H_0} \tag{2.5}$$

After introducing the complex quantities

$$h_x - ih_y = \varphi, \qquad u_x - iu_y = f + \frac{1}{R}\int_0^\tau q(s)ds, \qquad q_x - iq_y = q \tag{2.6}$$

the basic system (2.3) can be written in the more compact form

$$R\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \zeta^2} + \frac{M^2}{R_m}\frac{\partial \varphi}{\partial \zeta}, \qquad R_m\frac{\partial \varphi}{\partial \tau} = (1 + i\beta)\,\frac{\partial^2 \varphi}{\partial \zeta^2} + R_m\frac{\partial f}{\partial \zeta} \tag{2.7}$$

Applying a Laplace transformation to (2.7), with zero initial conditions, and using the notation

$$\Omega\ (\zeta,\ p) = \int_0^\infty \omega\ (\zeta,\ \tau)\ e^{-p\tau} d\tau \qquad (2.8)$$

we obtain the system

$$RpF = \frac{d^2F}{d\zeta^2} + \frac{M^2}{R_m}\frac{d\Phi}{d\zeta}, \qquad R_m p\Phi = (1 + i\beta)\frac{d^2\Phi}{d\zeta^2} + R_m\frac{dF}{d\zeta} \qquad (2.9)$$

the general solution of which can be written, for the symmetric case under consideration, in the form

$$F = -\frac{Q}{Rp}(A_1 \cosh\gamma_1\zeta + A_2 \cosh\gamma_2\zeta) \qquad (2.10)$$

$$\Phi = -\frac{QR_m}{M^2Rp}\left[A_1(Rp - \gamma_1^2)\frac{\sinh\gamma_1\zeta}{\zeta} + A_2(Rp - \gamma_2^2)\frac{\sinh\gamma_2\zeta}{\zeta}\right] \qquad (2.11)$$

$$\gamma_{1,2} = \frac{1}{2\sqrt{\alpha}}\left[\sqrt{M^2 + p(\sqrt{R\alpha} + \sqrt{R_m})^2} \pm \sqrt{M^2 + p(\sqrt{R\alpha} - \sqrt{R_m})^2}\right]$$

$$(\alpha = 1 + i\beta)$$

The quantities A_1 and A_2 have to be found from the boundary conditions on the channel walls, consisting of, first, equality of the velocity of the medium with the velocity of the planes bounding it, and, second, the requirement that the tangential components of the electric and magnetic fields be continuous from the gas into the wall. To simplify the results in what follows, the planes $z = \pm a$ are taken to be stationary, so that one of the boundary conditions has the form

$$F|_{\zeta=1} = -\frac{Q}{Rp} \qquad (2.12)$$

To obtain the second condition it is necessary to investigate Maxwell's equations in the region $\zeta > 1$, which, after Laplace transformation, have the form

$$\frac{d\Phi^*}{d\zeta} = R_m^*(E_y^* + iE_x^*), \qquad \frac{d}{d\zeta}(E_y^* + iE_x^*) = p\Phi^* \qquad (2.13)$$

(displacement currents are neglected, the index * refers to the region in the walls). Solving these equations with the condition that the fields be bounded for $\zeta \to \infty$, we find a relation between the electric and magnetic fields,

$$E_y^* + iE_x^* = -\sqrt{\frac{p}{R_m^*}}\,\Phi^* \qquad (\zeta > 1) \qquad (2.14)$$

Taking account of the continuity of the quantities E_x, E_y and Φ, as well as Equations (2.5), we obtain the second boundary condition

$$\left[\alpha \frac{d\Phi}{d\zeta} + R_m \sqrt{\frac{p}{R_m{}^*}}\, \Phi\right]_{\zeta=1} = 0 \qquad (\alpha = 1 + i\beta) \tag{2.15}$$

Putting (2.10) in (2.12) and (2.15), and solving the resulting algebraic system, we find

$$\begin{aligned} A_1 &= \frac{Rp - \gamma_2^2}{D}\left[\alpha \cosh \gamma_2 + R_m \sqrt{\frac{p}{R_m{}^*}} \frac{\sinh \gamma_2}{\gamma_2}\right] \\ A_2 &= -\frac{Rp - \gamma_1^2}{D}\left[\alpha \cosh \gamma_1 + R_m \sqrt{\frac{p}{R_m{}^*}} \frac{\sinh \gamma_1}{\gamma_1}\right] \\ D &= \alpha\,(\gamma_1^2 - \gamma_2^2) \cosh \gamma_1 \cosh \gamma_2 + R_m \sqrt{\frac{p}{R_m{}^*}}\left[\frac{Rp - \gamma_2^2}{\gamma_2} \cosh \gamma_1 \sinh \gamma_2 - \right. \\ &\quad \left. - \frac{Rp - \gamma_1^2}{\gamma_1} \cosh \gamma_2 \sinh \gamma_1\right] \end{aligned} \tag{2.16}$$

Thus, the general solution of the problem posed is given by the following complex integrals:

$$\begin{aligned} u_x - iu_y &= \frac{1}{R}\int_0^\tau q(s)\, ds + \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} F(\zeta, p) \exp(p\tau)\, dp \\ h_x - ih_y &= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Phi(\zeta, p) \exp(p\tau)\, dp \end{aligned} \tag{2.17}$$

where the functions under the integrals are determined from Equations (2.10) and (2.16).

For $\omega\tau^* = 0$ ($\alpha = 1$) the solution corresponds to flow with isotropic conductivity [6].

The solution obtained can be put into real form by representing the complex integrals (2.17) as the sum of residues at the poles corresponding to the roots of the equation $D(p) = 0$, together with the integral over $\mathrm{Re}\, p < 0$, $\mathrm{Im}\, p = 0$, which must be carried out in the presence of a branch point at $p = 0$. The boundary value problems of mathematical physics which correspond to the case under consideration have, as in the isotropic case [7], a mixed spectrum of eigenvalues.

In view of the complexity of the indicated computations, we investigate in the following only the particular case when the conductivity of the gas is not large.

3. Case of a weakly conducting gas. Let us assume that the conductivity of the gas σ is small compared to the conductivity of the walls σ^* and, in addition, the viscous Reynolds number is considerably greater than the magnetic Reynolds number. Assuming that the conditions

$$R_m \ll \sqrt{R_m{}^*}, \qquad \sqrt{R_m} \ll \sqrt{R} \tag{3.1}$$

are satisfied, expanding the exact solution in a power series of small parameters and taking only the first terms, we obtain an approximate solution of the problem in the form of complex integrals having the following form:

$$u_x - iu_y = \frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty} \frac{Q}{\gamma^2}\left(1-\frac{\cosh\gamma\zeta}{\cosh\gamma}\right)\exp(p\tau)\,dp$$

$$h_x - ih_y = \frac{R_m}{\alpha}\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b-i\infty} \frac{Q}{\gamma^2}\left(\frac{\sinh\gamma\zeta}{\gamma\cosh\gamma} - \zeta\right)\exp(p\tau)\,dp \qquad \left(\gamma = \sqrt{Rp + \frac{M^2}{\alpha}}\right) \tag{3.2}$$

The integrals may be evaluated, for arbitrary time dependence of the pressure gradient $q(\tau)$, with the help of the convolution theorem.

For example, we have for the velocity of the medium

$$u_x - iu_y = \int\limits_0^\tau q(\tau - u)\,\omega(u)\,du \tag{3.3}$$

where the function

$$\omega(\tau) = \frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty} \frac{\exp(p\tau)}{\gamma^2}\left(1-\frac{\cosh\gamma\zeta}{\cosh\gamma}\right)dp \tag{3.4}$$

is easily found with the help of the theorem of residues

$$\omega(\tau) = \frac{2}{R}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n}\exp\left[-\left(\lambda_n^2 + \frac{M^2}{\alpha}\right)\frac{\tau}{R}\right]\cos\lambda_n\zeta, \quad \lambda_n = \frac{2n+1}{2}\pi \tag{3.5}$$

Further computations will be carried out under the assumption that a steady pressure gradient P is imposed in the x-direction. Inasmuch as

$$q(\tau) = \frac{Pa^2}{U_0\eta} = q = \text{const}$$

therefore

$$u_x - iu_y = 2q\sum_{n=0}^{\infty}\frac{(-1)^n\cos\lambda_n\zeta}{\lambda_n\left(\lambda_n^2 + \frac{M^2}{\alpha}\right)}\left\{1 - \exp\left[-\left(\lambda_n^2 + \frac{M^2}{\alpha}\right)\frac{\tau}{R}\right]\right\} \tag{3.6}$$

Summing the series by means of the formula

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cos\lambda_n\zeta}{\lambda_n(\lambda_n^2 + c^2)} = \frac{\cosh c - \cosh c\zeta}{2c^2\cosh c} \qquad (0<\zeta<1) \tag{3.7}$$

we obtain the following expression

$$u_x - iu_y = \frac{q(1+i\beta)}{M^2}\left[1 - \frac{\cosh(M\zeta / \sqrt{1+i\beta})}{\cosh(M / \sqrt{1+i\beta})}\right] -$$
$$- 2q\exp\left[-\frac{M^2\tau}{R(1+i\beta)}\right]\sum_{n=0}^{\infty}\frac{(-1)^n\cos\lambda_n\zeta}{\lambda_n^2[\lambda_n^2 + M^2/(1+i\beta)]}\exp\left(-\frac{\lambda_n^2\tau}{R}\right) \qquad (3.8)$$

the first term of which represents the stationary regime of the flow under consideration, and previously found in [10].

Similar relations can also be found for the induced magnetic fields.

An investigation of expression (3.8) shows that, for $\beta \neq 0$ (anisotropic conductivity), the transient regime will contain a periodic function of time with frequency $M^2\beta/R(1 + \beta^2)$, while for $\beta = 0$ (isotropic conductivity) the approach to the stationary state has an aperiodic character.

Figure 2 shows the values of the gas flow W_x and W_y in the x- and y-directions, compared to the corresponding stationary values W_x° and W_y°, and computed from the relations

$$\Phi_x = \frac{W_x}{W_x^{\circ}} = 1 - \frac{\exp-(\lambda\theta)}{\sigma_1(0)}[\sigma_1(\theta)\cos\lambda\beta\theta - \sigma_2(\theta)\sin\lambda\beta\theta]$$
$$\Phi_y = \frac{W_y}{W_y^{\circ}} = 1 - \frac{\exp(-\lambda\theta)}{\sigma_2(0)}[\sigma_1(\theta)\sin\lambda\beta\theta + \sigma_2(\theta)\cos\lambda\beta\theta] \qquad (3.9)$$

where

$$\sigma_1(\theta) = \sum_{n=0}^{\infty}\frac{\lambda_n^2(1+\beta^2)+M^2}{[\lambda_n^2(1+\beta^2)+M^2]^2+M^4\beta^2}\frac{1}{\lambda_n^2}\exp\left(-\lambda_n^2\frac{\theta}{M^2}\right) \qquad (3.10)$$
$$\sigma_2(\theta) = \sum_{n=0}^{\infty}\frac{\beta M^2}{[\lambda_n^2(1+\beta^2)+M^2]^2+M^4\beta^2}\frac{1}{\lambda_n^2}\exp\left(-\lambda_n^2\frac{\theta}{M^2}\right)$$

Here the notation is

$$\theta = \frac{M^2\tau}{R}, \qquad \lambda = \frac{1}{1+\beta^2} \qquad (3.11)$$

Computations were carried out for two values of the Hartmann number, $M = 5$ and $M = 10$, with the anisotropy parameter β being taken as 3. Also shown on Fig. 2 are the results for the case $M = \infty$, corresponding to the flow of an inviscid gas; in that case Equations (3.9) go over to the corresponding equations of [9].

The numerical results obtained clearly illustrate the influence of viscosity on the form of the damped oscillations comprising the transient regime: the first maxima of amplitude are reached for the same values of

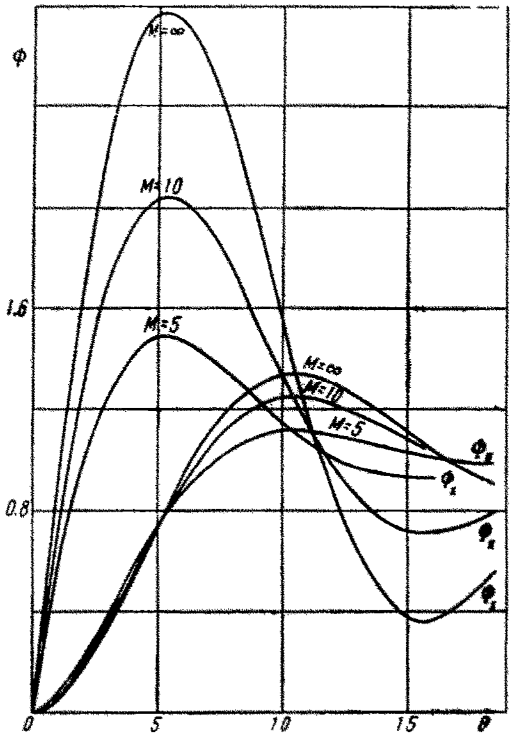

Fig. 2.

time, but their magnitude decreases with increasing coefficient of viscosity; the latter effect is particularly significant for the discharge in the direction of the applied pressure gradient.

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Translated by A.R.